Entanglement cost and distillable entanglement of symmetric states

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February 1, 2008

Abstract

We compute entanglement cost and distillable entanglement of states supported on symmetric subspace. Not only giving general formula, we apply them to the output states of optimal cloning machines. Surprisingly, under some settings, the optimal n to m clone and true m copies are the same in entanglement measures. However, they differ in the error exponent of entanglement dilution. We also presented a general theory of entanglement dilution which is applicable to any non-i.i.d sequence of states.

1 Introduction

In asymptotic theory of entanglement,it is often assumed that the given state is in the form of $\rho^{\otimes n}$, or independent identical copies of a state ρ (i.i.d. ensemble, hereafter). In some important cases, however, this assumption is not necessarily true. For example, in study of local copying [1][13], we have to treat the optimal clone of a bipartite state: Given n copies of them, its optimal n to m clone is not close to i.i.d ensemble at all. The purpose of this manuscript is to give explicit, tractable formula of entanglement cost and distillable entanglement of non-i.i.d. states.

In the manuscript, we discuss entanglement cost of general non-i.i.d. state, using information spectrum [4][5][6][10]. (in quantum information jargon, it is called smooth Reny entropy.) This formula, however, contains maximization which cannot be solved in most of the cases.

Therefore, second, we present a formula without maximization for symmetric states, or states supported on symmetric subspace. For such states, we also give distillable entanglement, too. Remarkably, the optimal entangle distillation is

possible without knowing the input state except the fact that it is a symmetric state. This is a generalization of universal entanglement concentration in [12].

Finally, we apply this theory to output states of n to m cloning machines. We assume that the input is n copies of the identical pure states, and m equals rn for some constant r. The following two kinds of cloning machines are considered.

First example is the machine optimal for the case where the Schmidt basis of the input is known except its phases. To our surprise, both the entanglement cost and distillable entanglement are m times entropy of entanglement of the input state. Hence, optimal clone and real m copies are the same in its entanglement measures. We also computed error exponent of entanglement distillation and dilution, and showed they are worse than real m copies.

Second example is the machine optimal for all the possible pure bipartite states. For this, we only proved that m times the entropy of entanglement cost of the input state is an upperbound of the entanglement cost. Our conjecture is that this upperbound is the entanglement cost and, at the same time, the distillable entanglement.

2 A general theory of entanglement cost

Below, $|\Phi_D\rangle$ is a maximally entangled state with Schmidt rank D, and $\mathrm{F}(\rho, |\Phi_D\rangle)$ is the optimal fidelity of generating ρ from $|\Phi_D\rangle$ by LOCC.

Lemma 1

$$\mathrm{F}^D(\rho) = \max_{\{q_i, |\phi_i\rangle\}} \sum_i \sum_{j=1}^{D} q_i p_j^{\phi_i},$$

where $p_j^{\phi_i}$ is the jth largest Schmidt coefficient of $|\phi_i\rangle$, and the maximization is taken over pure state ensembles with $\sum_i q_i |\phi_i\rangle\langle\phi_i| = \rho$.

Proof. Observe

$$\begin{aligned}\mathrm{F}^D(\rho) &= \max_{\{A_i\}} \mathrm{F}\left(\rho, \sum_j A_j |\Phi_D\rangle\langle\Phi_D| A_j^\dagger\right) \\ &= \max_{\{A_i\}} \max_{\{q_i, |\phi_i\rangle\}} \left|\sum_{i,j} \sqrt{q_i} \langle\phi_i| A_i |\Phi_D\rangle\right|^2.\end{aligned}$$

We solve maximization over $\{A_i\}$. Since they are LOCC, the Schmidt rank of $A_i|\Phi_D\rangle$ cannot be more than D. Therefore, it is optimal if

$$A_i |\Phi_D\rangle = \frac{c_i}{\sqrt{\sum_{j=1}^{D} p_j^{\phi_i}}} \sum_{j=1}^{D} \sqrt{p_j^{\phi_i}} |j\rangle |j\rangle,$$

and this is possible for any $\{c_i\}$ with $\sum_i |c_i|^2 = 1$. Therefore,

$$
\begin{aligned}
\mathrm{F}^D(\rho) &= \max_{\{q_i,|\phi_i\rangle\}} \max_{\{c_i\}:\sum_i |c_i|^2=1} \left| \sum_i \sqrt{q_i} \frac{c_i}{\sqrt{\sum_{j=1}^D p_j^{\phi_i}}} \sum_{j=1}^D p_j^{\phi_i} \right|^2 \\
&= \max_{\{q_i,|\phi_i\rangle\}} \max_{\{c_i\}:\sum_i |c_i|^2=1} \left(\sum_i c_i \sqrt{q_i \sum_{j=1}^D p_j^{\phi_i}} \right)^2 \\
&= \max_{\{q_i,|\phi_i\rangle\}} \sum_i q_i \sum_{j=1}^D p_j^{\phi_i}.
\end{aligned}
$$

■

Given a sequence $\{\rho^n\}_{n=1}^{\infty}$ of bipartite quantum states, we consider a sequence of purestate ensembles $\{q_i^n, |\phi_i^n\rangle\}$ with $\sum_i q_i^n |\phi_i^n\rangle \langle\phi_i^n| = \rho^n$. Let $p_j^{n,i}$, where j runs from 1 to the Schmidt rank of $|\phi_i^n\rangle$, be the Schmidt coefficients of $|\phi_i^n\rangle$ ($p_1^{n,i} \geq p_2^{n,i} \cdots$). Then, $q_i^n p_j^{n,i}$ defines a probability distribution over (i,j). At the same time, the value $p_j^{n,i}$ can be viewed as a random variable, where (i,j) occurs with the probability $q_i^n p_j^{n,i}$.

Given a sequence of probability distributions $\{P^n\}_{n=1}^{\infty}$ over some discrete set, we define a notion of probabilistic limsup of a random variable X^n, denote by $\mathrm{p}-\overline{\lim}_{n\to\infty} X^n$, the minimum of x with

$$
\lim_{n\to\infty} P^n \{ i \; ; \; X^n \leq x \} = 1.
$$

We also denote by $\mathrm{p}-\underline{\lim}_{n\to\infty} X^n$, the maximum of x with

$$
\lim_{n\to\infty} P^n \{ i \; ; \; X^n \geq x \} = 1.
$$

Theorem 2 *Given a sequence $\{\rho^n\}_{n=1}^{\infty}$ of bipartite quantum states, we have*

$$
E_c\left(\{\rho^n\}_{n=1}^{\infty}\right) = \inf_{\{q_i^n,|\phi_i^n\rangle\}} \mathrm{p}-\overline{\lim_{n\to\infty}} \frac{-1}{n} \log p_j^{n,i}
$$

where $\mathrm{p}-\overline{\lim}_{n\to\infty}$ is with respect to $\left\{q_i^n p_j^{n,i}\right\}_{n=1}^{\infty}$, and infimum is taken over all the sequences of pure state ensembles $\{q_i^n, |\phi_i^n\rangle\}$ with $\sum_i q_i^n |\phi_i^n\rangle \langle\phi_i^n| = \rho^n$.

Proof. We use the technique which repeatedly used in [4]. "$\leq$" is proved as follows. For any j_0, we have

$$
1 \geq \sum_{j=1}^{j_0} p_j^{n,i} \geq j_0 p_{j_0}^{n,i},
$$

implying

$$\left\{ j : p_j^{n,i} \geq c^{-1} \right\} \subset \{ j : j \leq c \} ,$$

and

$$\mathfrak{C}_R^n \subset \mathfrak{D}_R^n, \tag{1}$$

where

$$\mathfrak{C}_R^n := \left\{ (i,j) ;\ p_j^{n,i} \geq 2^{-nR} \right\},$$
$$\mathfrak{D}_R^n := \left\{ (i,j) ;\ j \leq 2^{nR} \right\}.$$

Therefore,

$$\sum_{(i,j) \in \mathfrak{C}_R^n} q_i^n p_j^{n,i} \leq \sum_{(i,j) \in \mathfrak{D}_R^n} q_i^n p_j^{n,i} \leq \mathrm{F}^{2^{nR}} \left(\rho^{\otimes n} \right). \tag{2}$$

If $R > \text{p}-\overline{\lim}_{n\to\infty} \frac{-1}{n} \log p_j^{n,i}$, the left most side tends to 1 as $n \to \infty$, meaning that

$$E_c \left(\{ \rho^n \}_{n=1}^{\infty} \right) \leq \text{p}-\overline{\lim_{n\to\infty}} \frac{-1}{n} \log p_j^{n,i}$$

holds for any pure state ensembles $\{ q_i^n, |\phi_i^n\rangle \}$ with $\sum_i q_i^n |\phi_i^n\rangle \langle \phi_i^n| = \rho^n$, and we have "$\leq$". "$\geq$" is proved as follows. Since

$$\overline{\mathfrak{C}_{R+\gamma}^n} \subset \left(\overline{\mathfrak{C}_{R+\gamma}^n} \cap \mathfrak{D}_R^n \right) \cup \overline{\mathfrak{D}_R^n},$$

we have

$$\sum_{(i,j) \in \overline{\mathfrak{C}_{R+\gamma}^n}} q_i^n p_j^{n,i} \leq \sum_{(i,j) \in \overline{\mathfrak{C}_{R+\gamma}^n} \cap \mathfrak{D}_R^n} q_i^n p_j^{n,i} + \sum_{(i,j) \in \overline{\mathfrak{D}_R^n}} q_i^n p_j^{n,i}.$$

Since $|\mathfrak{D}_R^n| = 2^{nR}$,

$$\begin{aligned} \sum_{(i,j) \in \overline{\mathfrak{C}_{R+\gamma}^n} \cap \mathfrak{D}_R^n} q_i^n p_j^{n,i} &\leq \sum_{(i,j) \in \mathfrak{D}_R^n} q_i^n 2^{-n(R+\gamma)} \\ &\leq \sum_i q_i^n 2^{nR} \cdot 2^{-n(R+\gamma)} \\ &= 2^{-n\gamma}. \end{aligned}$$

Hence, with a proper choice of pure state ensemble $\{ q_i^n, |\phi_i^n\rangle \}$, for any $\epsilon > 0$,

$$\begin{aligned} 1 - \mathrm{F}^{2^{nR}} (\rho^n) + \epsilon &\geq 1 - \sum_{(i,j) \in \mathfrak{D}_R^n} q_i^n p_j^{n,i} \\ &= \sum_{(i,j) \in \overline{\mathfrak{D}_R^n}} q_i^n p_j^{n,i} \\ &\geq \sum_{(i,j) \in \overline{\mathfrak{C}_{R+\gamma}^n}} q_i^n p_j^{n,i} - \sum_{(i,j) \in \overline{\mathfrak{C}_{R+\gamma}^n} \cap \mathfrak{D}_R^n} q_i^n p_j^{n,i} \\ &\geq \sum_{(i,j) \in \overline{\mathfrak{C}_{R+\gamma}^n}} q_i^n p_j^{n,i} - 2^{-n\gamma}. \end{aligned} \tag{3}$$

Suppose

$$
\begin{aligned}
R &< \inf_{\left[\left\{q_i^n, \left|\phi_i^n\right\rangle\right\}\right]_{n=1}^{\infty}} \text{p-} \overline{\lim_{n\to\infty}} \frac{-1}{n} \log p_j^{n,i}, \\
&\leq \text{p-} \overline{\lim_{n\to\infty}} \frac{-1}{n} \log p_j^{n,i}.
\end{aligned}
$$

Then, we can choose $\gamma > 0$ with there is $R + \gamma < \text{p-} \overline{\lim}_{n\to\infty} \frac{-1}{n} \log p_j^{n,i}$, so that the last end of this inequality does not vanish as $n \to \infty$. Hence, we cannot do entanglement dilution with high fidelity, if $R < \text{p-} \overline{\lim}_{n\to\infty} \frac{-1}{n} \log p_j^{n,i}$. Therefore, we have "$\geq$". ■

Theorem 3 *Suppose*

$$
R < \inf_{\left[\left\{q_i^n, \left|\phi_i^n\right\rangle\right\}\right]_{n=1}^{\infty}} \text{p-} \underline{\lim_{n\to\infty}} \frac{-1}{n} \log p_j^{n,i},
$$

where $\text{p-} \underline{\lim}_{n\to\infty}$ *is with respect to* $\left\{q_i^n p_j^{n,i}\right\}_{n=1}^{\infty}$*, and infimum is taken over all the sequences of pure state ensembles* $\{q_i^n, |\phi_i^n\rangle\}$ *with* $\sum_i q_i^n |\phi_i^n\rangle \langle\phi_i^n| = \rho^n$*. Then,*

$$
\mathrm{F}^{2^{nR}}(\rho^n) \to 0.
$$

Also, if

$$
R > \inf_{\left[\left\{q_i^n, \left|\phi_i^n\right\rangle\right\}\right]_{n=1}^{\infty}} \text{p-} \underline{\lim_{n\to\infty}} \frac{-1}{n} \log p_j^{n,i},
$$

$$
\overline{\lim_{n\to\infty}} \mathrm{F}^{2^{nR}}(\rho^n) > 0.
$$

Proof. We use the inequality (3). With a proper choice of pure state ensemble $\{q_i^n, |\phi_i^n\rangle\}$, for any $\epsilon > 0$,

$$
1 - \mathrm{F}^{2^{nR}}(\rho^n) + \epsilon \geq \sum_{(i,j)\in \overline{\mathfrak{C}_{R+\gamma}^n}} q_i^n p_j^{n,i} - 2^{-n\gamma}.
$$

Choose γ with

$$
\begin{aligned}
R + \gamma &< \inf_{\{q_i^n, |\phi_i^n\rangle\}} \text{p-} \underline{\lim_{n\to\infty}} \frac{-1}{n} \log p_j^{n,i} \\
&\leq \text{p-} \underline{\lim_{n\to\infty}} \frac{-1}{n} \log p_j^{n,i}.
\end{aligned}
$$

Then, due the definition of probabilistic liminf,

$$
\lim_{n\to\infty} \sum_{(i,j)\in \overline{\mathfrak{C}_{R+\gamma}^n}} q_i^n p_j^{n,i} = 1.
$$

Therefore,

$$\begin{aligned}\overline{\lim_{n\to\infty}}\, \mathrm{F}^{2^{nR}}\left(\rho^{\otimes n}\right) &\leq \lim_{n\to\infty} 2^{-n\gamma}+\epsilon \\ &= \epsilon.\end{aligned}$$

Since ϵ is arbitrary positive number, our first assertion is proved. Next, due to (2), we have

$$\sum_{(i,j)\in\mathfrak{C}^n_R} q_i^n p_j^{n,i} \leq \mathrm{F}^{2^{nR}}\left(\rho^n\right).$$

Due to the definition of probabilistic liminf, if $R > \inf_{[\{q_i^n,|\phi_i^n\rangle\}]_{n=1}^{\infty}} \text{p}-\underline{\lim}_{n\to\infty} \frac{-1}{n}\log p_j^{n,i}$, liminf of the left hand side does not vanish. Hence, we have our second assertion. ■

3 A standard form of symmetric states

Below, we discuss entanglement of symmetric states, or states supported on the symmetric subspace of $(\mathcal{H}_A\otimes\mathcal{H}_B)^{\otimes n}$, where $\mathcal{H}_A\simeq\mathcal{H}_B\simeq\mathcal{H}$ and $\dim\mathcal{H}=d$. For that purpose, we introduce a standard form of such states in this section. It suffices to give a standard form for a pure symmetric state, since a mixed state is convex combination of them.

Suppose we are given n-copies of unknown pure bipartite state $|\phi\rangle\in\mathcal{H}_A\otimes\mathcal{H}_B$, which is unknown. Here we assume $\mathcal{H}_A\simeq\mathcal{H}_B\simeq\mathcal{H}$ and $\dim\mathcal{H}=d$.

It is known that $|\phi\rangle^{\otimes n}$, where $|\phi\rangle\in\mathcal{H}_A\otimes\mathcal{H}_B$, has the standard form defined as follows. Note $|\phi\rangle^{\otimes n}$ is invariant by the reordering of copies, or the action of the permutation σ in the set $\{1,\dots n\}$ such that

$$\bigotimes_{i=1}^n |h_{i,A}\rangle|h_{i,B}\rangle \mapsto \bigotimes_{i=1}^n |h_{\sigma^{-1}(i),A}\rangle|h_{\sigma^{-1}(i),B}\rangle, \tag{4}$$

where $|h_{i,A}\rangle\in\mathcal{H}_A$ and $|h_{i,B}\rangle\in\mathcal{H}_B$. Action of the symmetric group occurs a decomposition of the tensored space $\mathcal{H}^{\otimes n}$ [16],

$$\mathcal{H}^{\otimes n}=\bigoplus_\lambda \mathcal{W}_\lambda,\ \mathcal{W}_\lambda:=\mathcal{U}_\lambda\otimes\mathcal{V}_\lambda.$$

Here, $\mathcal{U}_\lambda$ and $\mathcal{V}_\lambda$ is an irreducible space of the tensor representation of SU(d), and the representation (4) of the symmetric group, respectively, and

$$\lambda=(\lambda_1,\dots,\lambda_d),\qquad \lambda_i\geq\lambda_{i+1}\geq 0,\ \sum_{i=1}^d\lambda_i=n$$

is called *Young index*, which $\mathcal{U}_\lambda$ and $\mathcal{V}_\lambda$ uniquely corresponds to. To emphasize $\sum_{i=1}^d\lambda_i=n$, we use the notation " $\lambda\vdash n$". We denote by $\mathcal{U}_{\lambda,A}$, $\mathcal{V}_{\lambda,A}$, and $\mathcal{U}_{\lambda,B}$, $\mathcal{V}_{\lambda,B}$ the irreducible component of $\mathcal{H}_A^{\otimes n}$ and $\mathcal{H}_B^{\otimes n}$, respectively. Also, $\mathcal{W}_{\lambda,A}:=\mathcal{U}_{\lambda,A}\otimes\mathcal{V}_{\lambda,A}$, $\mathcal{W}_{\lambda,B}:=\mathcal{U}_{\lambda,B}\otimes\mathcal{V}_{\lambda,B}$.

In terms of this decomposition, $|\phi\rangle^{\otimes n}$ can be written as

$$|\phi\rangle^{\otimes n} = \bigoplus_{\lambda:\lambda\vdash n} a_\lambda \left|\phi_\lambda\right\rangle \left|\Phi_\lambda\right\rangle ,$$

where $|\phi_\lambda\rangle \in \mathcal{U}_{\lambda,A} \otimes \mathcal{U}_{\lambda,B}$, and $|\Phi_\lambda\rangle \in \mathcal{V}_{\lambda,A} \otimes \mathcal{V}_{\lambda,B}$. While a_λ and $|\phi_\lambda\rangle$ are dependent on $|\phi\rangle$, $|\Phi_\lambda\rangle$ is a maximally entangled state which does not depend on $|\phi\rangle$,

$$|\Phi_\lambda\rangle := \frac{1}{\sqrt{d_\lambda}} \sum_{i=1}^{d_\lambda} |f_i\rangle \, |f_i\rangle ,$$

with $\{|f_i\rangle\}$'s being an orthonormal complete basis of $\mathcal{V}_\lambda$, and $d_\lambda := \dim \mathcal{V}_\lambda$.

Therefore, any symmetric pure state, being a superposition of n-tensored pure states, can be written as

$$\bigoplus_{\lambda:\lambda\vdash n} a_\lambda \left|\phi_\lambda\right\rangle \left|\Phi_\lambda\right\rangle \ . \tag{5}$$

4 Entanglement cost of symmetric states

For Young indices $\lambda \vdash n$ and l with $1 \leq l \leq d_\lambda$, let

$$b^n_{\lambda l} = \frac{\mathrm{tr}\, \rho^n \mathcal{W}_{\lambda,A} \otimes \mathbf{1}_B}{d_\lambda}.$$

Note $b^n_{\lambda l}$ does not vary with l. Note also $b^n_{\lambda l}$ defines a probability distribution over (λ, k).

Lemma 4 *If the state is supported on the symmetric subspace of* $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$,

$$\mathrm{p-}\overline{\lim_{n\to\infty}} \frac{-1}{n} \log b^n_{\lambda l} \leq E_c\left(\{\rho^n\}_{n=1}^{\infty}\right), \tag{6}$$

where the underlying sequence of probability measure is $\{b^n_{\lambda k}\}_{n=1}^{\infty}$.

Proof. A composition of local projective measurement $\{\mathcal{W}_{\lambda,A} \otimes \mathcal{W}_{\lambda,B}\}_\lambda$ followed by tracing out $\mathcal{U}_A \otimes \mathcal{U}_B$ sends symmetric state ρ^n, a convex combination of a state in the form of (5), to

$$\sigma^n := \bigoplus_{\lambda:\lambda\vdash n} \sum_{l=1}^{d_\lambda} b^n_{\lambda l} |\Phi_\lambda\rangle \langle\Phi_\lambda|, \tag{7}$$

where $|\Phi_\lambda\rangle$ is a maximally entangled state living in $\mathcal{V}_{\lambda,A} \otimes \mathcal{V}_{\lambda,B}$. Since this operation is LOCC, dilution of σ^n is easier than $\rho^{n,m}_\phi$, and

$$\mathrm{F}^{2^{nR}}\left(\sigma^n\right) \geq \mathrm{F}^{2^{nR}}\left(\rho^n\right). \tag{8}$$

Let

$$\mathfrak{E}_R^n := \left\{ (\lambda, l)\,;\, b_{\lambda l}^n \geq 2^{-nR} \right\},$$
$$\mathfrak{F}_R^n := \left\{ (\lambda, l)\,;\, d_\lambda \leq 2^{nR} \right\}.$$

Observe

$$\mathrm{F}^{2^{nR}}(\sigma^n) = \sum_{(\lambda,l)\in\mathfrak{F}_{R^n}^n} b_{\lambda k}^n,$$

where R^n is decided by

$$R^n = \max\left\{ S\,;\, \sum_{\lambda\,:\,d_\lambda \leq 2^{nS}} d_\lambda \leq 2^{nR} \right\}. \tag{9}$$

Here, note that R^n is a function of R, though we don't write it explicitly. Since

$$2^{nS} \leq \sum_{\lambda\,:\,d_\lambda \leq 2^{nS}} d_\lambda \leq (n+1)^d\, 2^{nS},$$

we have

$$R - \frac{d}{n}\log(n+1) \leq R^n \leq R. \tag{10}$$

Since

$$\mathfrak{F}_{R^n}^n \subset \mathfrak{F}_R^n \subset \left(\mathfrak{F}_R^n \cap \overline{\mathfrak{E}_{R+\gamma}^n} \right) \cup \mathfrak{E}_{R+\gamma}^n$$

holds,

$$\sum_{(\lambda,l)\in\mathfrak{F}_{R^n}^n} b_{\lambda l}^n \leq \sum_{(\lambda,l)\in\mathfrak{F}_R^n\cap\overline{\mathfrak{E}_{R+\gamma}^n}} b_{\lambda l}^n + \sum_{(\lambda,l)\in\mathfrak{E}_{R+\gamma}^n} b_{\lambda l}^n.$$

We show the first term of the right hand side is negligible for any $\gamma > 0$:

$$\begin{aligned}
\sum_{(\lambda,l)\in\mathfrak{F}_R^n\cap\overline{\mathfrak{E}_R^n}} b_{\lambda k}^n &\leq 2^{-n(R+\gamma)} \left| \left\{ (\lambda, l)\,;\, d_\lambda \leq 2^{nR^n} \right\} \right| \\
&= 2^{-n(R+\gamma)} \sum_{\lambda: d_\lambda \leq 2^{nR}} d_\lambda \\
&\leq 2^{-n(R+\gamma)} \cdot 2^{nR} \cdot (n+1)^d \\
&\leq (n+1)^d\, 2^{-n\gamma}.
\end{aligned}$$

Therefore,

$$\mathrm{F}^{2^{nR}}(\sigma^n) \leq \sum_{(\lambda,l)\in\mathfrak{E}_{R+\gamma}^n} b_{\lambda l}^n + (n+1)^d\, 2^{-n\gamma}. \tag{11}$$

which, combined with (8), implies

$$\begin{aligned}E_c\left(\{\rho^n\}_{n=1}^{\infty}\right) &= \inf\left\{R\ ;\ \lim_{n\to\infty}\mathrm{F}^{2^{nR}}\left(\rho^n\right)=1\right\}\\ &\geq \inf\left\{R\ ;\ \lim_{n\to\infty}\mathrm{F}^{2^{nR}}\left(\sigma^n\right)=1\right\}\\ &\geq \inf\left\{R\ ;\ \lim_{n\to\infty}\sum_{(\lambda,l)\in\mathfrak{E}^n_{R+\gamma}} b^n_{\lambda l}+(n+1)^d\,2^{-n\gamma}=1\right\}\\ &= \inf\left\{R\ ;\ \lim_{n\to\infty}\sum_{(\lambda,l)\in\mathfrak{E}^n_{R+\gamma}} b^n_{\lambda l}=1\right\}\\ &= \mathrm{p}\!-\overline{\lim_{n\to\infty}}\frac{-1}{n}\log b^n_{\lambda l}-\gamma.\end{aligned}$$

Since this holds for any $\gamma>0$, our assertion is proved. ■

Below, we present a dilation protocol achieving the left hand side of (6). First, Bob fabricates the state locally, and applies the binary projective measurement

$$\left\{\sum_{\lambda:d_\lambda\leq 2^{nR}}\mathcal{W}_{\lambda,B},\ \mathbf{1}-\sum_{\lambda:d_\lambda\leq 2^{nR}}\mathcal{W}_{\lambda,B}\right\},$$

where $R=\mathrm{p}-\overline{\lim}_{n\to\infty}\frac{-1}{n}\log b^n_{\lambda l}+\gamma$ $(\gamma>0)$. If the event corresponding to $\sum_{\lambda:d_\lambda\leq 2^{nR}}\mathcal{W}_{\lambda,B}$ is observed, he teleports the part which should belong to Alice.

This procedure consumes the following amount of entanglement:

$$\begin{aligned}&\log\sum_{d_\lambda\leq 2^{nR}} d_\lambda \dim\mathcal{U}_\lambda\\ &\leq nR+d\log(n+1)+d^2\log n,\end{aligned}$$

(see (24)). Dividing both ends by n and taking $\overline{\lim}_{n\to\infty}$, the left hand side becomes R, which can be arbitrarily close to $\mathrm{p}-\overline{\lim}_{n\to\infty}\frac{-1}{n}\log b^n_{\lambda l}$. The success fidelity of this protocol is

$$\sum_{(\lambda,l)\in\mathfrak{F}^n_R} b^n_{\lambda l}\,.$$

Since $1\geq d_\lambda b^n_{\lambda l}$ or $d_\lambda\leq (b^n_{\lambda l})^{-1}$, we have $\mathfrak{F}^n_R\supset\mathfrak{E}^n_R$ and

$$\sum_{(\lambda,l)\in\mathfrak{F}^n_R} b^n_{\lambda l}\geq\sum_{(\lambda,l)\in\mathfrak{E}^n_R} b^n_{\lambda l},\tag{12}$$

which tends to 1 since $R>\mathrm{p}-\overline{\lim}_{n\to\infty}\frac{-1}{n}\log b^n_{\lambda l}$. Therefore, combined with lemma 4, we have proved:

Theorem 5 *If ρ^n is a symmetric state,*

$$E_c\left(\{\rho^n\}_{n=1}^{\infty}\right) = \mathrm{p}-\overline{\lim_{n\to\infty}}\frac{-1}{n}\log b_{\lambda l}^n,$$

where the underlying sequence of probability measure is $\{b_{\lambda l}^n\}_{n=1}^{\infty}$. $E_c\left(\{\rho^n\}_{n=1}^{\infty}\right)$ can be achieved by creating state locally and teleporting it.

Below, we derive another expression of $E_c\left(\{\rho^n\}_{n=1}^{\infty}\right)$. Let $\{c_{\lambda\,k\,l}^n\}$ be spectrum of the reduced density matrix $\mathrm{tr}_{\mathcal{H}_A}\rho^n$.

$$\mathrm{tr}_{\mathcal{H}_A}\rho^n = \sum_{\lambda,\,k,\,l} c_{\lambda\,k\,l}^n\,|\lambda\,k\,l\rangle\,\langle\lambda\,k\,l\,|\,,$$

where $|\lambda\,k\,l\rangle \in \mathcal{W}_\lambda$, and the indices k and l corresponds to the freedom of $\mathcal{U}_\lambda$ and $\mathcal{V}_\lambda$, respectively. Note that the Schmidt coefficient $c_{\lambda kl}^n$ does not depend on l, and that

$$\sum_{k=1}^{\dim\mathcal{U}_\lambda} c_{\lambda kl}^n = b_{\lambda l}^n\,.$$

Theorem 6

$$\mathrm{p}-\overline{\lim_{n\to\infty}}\frac{-1}{n}\log c_{\lambda kl}^n = E_c\left(\{\rho^n\}_{n=1}^{\infty}\right),$$

where the probabilistic limsup is with respect to the sequence of probability measure $\{c_{\lambda kl}^n\}_{n=1}^{\infty}$.

Proof. Due to the definition of probabilistic limsup,

$$\begin{aligned}
E_c\left(\{\rho^n\}_{n=1}^{\infty}\right) &= \mathrm{p}-\overline{\lim_{n\to\infty}}\frac{-1}{n}\log\sum_{k=1}^{\dim\mathcal{U}_\lambda} c_{\lambda kl}^n \quad (\text{w.r.t. } \{b_{\lambda l}^n\}) \qquad (13)\\
&= \mathrm{p}-\overline{\lim_{n\to\infty}}\frac{-1}{n}\log\sum_{k'=1}^{\dim\mathcal{U}_\lambda} c_{\lambda k' l}^n \quad (\text{w.r.t. } \{c_{\lambda kl}^n\})\\
&\le \mathrm{p}-\overline{\lim_{n\to\infty}}\frac{-1}{n}\log c_{\lambda kl}^n \quad (\text{w.r.t. } \{c_{\lambda kl}^n\}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathrm{p}-\overline{\lim_{n\to\infty}}\frac{-1}{n}\log c_{\lambda kl}^n \;\; (\text{w.r.t. } \{c_{\lambda kl}^n\})\\
&\le \mathrm{p}-\overline{\lim_{n\to\infty}}\frac{-1}{n}\log b_{\lambda l}^n \; (\text{w.r.t. } \{c_{\lambda kl}^n\})\\
&+ \mathrm{p}-\overline{\lim_{n\to\infty}}\frac{-1}{n}\log\left(c_{\lambda kl}^n / b_{\lambda l}^n\right) \;\; (\text{w.r.t. } \{c_{\lambda kl}^n\}).
\end{aligned}$$

Observe that $\{c^n_{\lambda kl}/b^n_{\lambda l}\}_{k=1}^{\dim \mathcal{U}_\lambda}$ defines a probability distribution over k $(1 \le k \le \dim \mathcal{U}_\lambda)$. Letting R and γ be an arbitrary positive real number, we have

$$
\begin{aligned}
&\sum_{\lambda,k,l\,:\,\left(c^n_{\lambda kl}/b^n_{\lambda l}\right)\ge 2^{-n(R+\gamma)}} c^n_{\lambda kl} \\
&= \sum_{\lambda,l}\left[b^n_{\lambda l} \sum_{k\,:\,\left(c^n_{\lambda kl}/b^n_{\lambda l}\right)\ge 2^{-n(R+\gamma)}} \frac{c^n_{\lambda kl}}{b^n_{\lambda l}} \right] \\
&\ge \sum_{\lambda,l}\left[b^n_{\lambda l} \sum_{k\,:\,k\le 2^{nR},\,\left(c^n_{\lambda kl}/b^n_{\lambda l}\right)\ge 2^{-n(R+\gamma)}} \frac{c^n_{\lambda kl}}{b^n_{\lambda l}} \right] \\
&= \sum_{\lambda,l}\left[b^n_{\lambda l} \left\{ \sum_{k\,:\,k\le 2^{nR}} \frac{c^n_{\lambda kl}}{b^n_{\lambda l}} - \sum_{k\,:\,k\le 2^{nR},\,\left(c^n_{\lambda kl}/b^n_{\lambda l}\right)< 2^{-n(R+\gamma)}} \frac{c^n_{\lambda kl}}{b^n_{\lambda l}} \right\} \right] \\
&\ge \sum_{\lambda,l}\left[b^n_{\lambda l} \left\{ \sum_{k\,:\,k\le 2^{nR}} \frac{c^n_{\lambda kl}}{b^n_{\lambda l}} - \sum_{k\,:\,k\le 2^{nR}} 2^{-n(R+\gamma)} \right\} \right] \\
&= \sum_{\lambda,l} b^n_{\lambda l} \sum_{k\,:\,k\le 2^{nR}} \frac{c^n_{\lambda kl}}{b^n_{\lambda l}} - 2^{-n\gamma}.
\end{aligned}
$$

Since $2^{nR} \ge \dim \mathcal{U}_\lambda$ holds for any $R > 0$ with large n, the last end of the inequality converges to 1. Hence, the left most end converges to 1, also. This means

$$
\text{p-}\varlimsup_{n\to\infty} \frac{-1}{n} \log\left(c^n_{\lambda kl}/b^n_{\lambda l}\right) \ \left(\text{w.r.t. } \{c^n_{\lambda kl}\}\right) \le R+\gamma.
$$

Letting $R \to 0$ and $\gamma \to 0$, we obtain

$$
\text{p-}\varlimsup_{n\to\infty} \frac{-1}{n} \log\left(c^n_{\lambda kl}/b^n_{\lambda l}\right) \ \left(\text{w.r.t. } \{c^n_{\lambda kl}\}\right) = 0 \tag{14}
$$

and

$$
\begin{aligned}
&\text{p-}\varlimsup_{n\to\infty} \frac{-1}{n} \log c^n_{\lambda kl} \le \text{p-}\varlimsup_{n\to\infty} \frac{-1}{n} \log b^n_{\lambda l} \\
&= E_c\left(\{\rho^n\}_{n=0}^{\infty}\right).
\end{aligned}
$$

Combining this with (13), we have the assertion. ■

5 Distillable entanglement of symmetric states

Lemma 7 *If ρ^n is a symmetric state,*

$$
E_d\left(\{\rho^n\}_{n=0}^{\infty}\right) \ge \text{p-}\varliminf_{n\to\infty} \frac{-1}{n} \log b^n_{\lambda l}\ ,
$$

where the probabilistic limsup is with respect to the sequence of probability measure $\{b^n_{\lambda l}\}_{n=1}^{\infty}$. Especially, the right hand side can be achieved without knowing ρ^n, expect the fact that it is a symmetric state.

Proof. Alice and Bob applies $\{\mathcal{W}_{\lambda,A}\}_\lambda$ and $\{\mathcal{W}_{\lambda,B}\}_\lambda$ independently, and trace out $\mathcal{U}_{\lambda,A}$ and $\mathcal{U}_{\lambda,B}$, respectively. Then, they obtain $\frac{1}{n}\log d_\lambda$ ebits of Bell pairs with the probability $d_\lambda b^n_{\lambda l}$. They also obtain classical information about λ, so they exactly know the shared entangled state. Obviously, this protocol can be implemented without knowing the input. Obviously, the yield of the protocol is

$$\text{p}-\varliminf_{n\to\infty}\frac{1}{n}\log d_\lambda=\sup_R\left\{R\ ;\ \lim_{n\to\infty}\sum_{(\lambda,l)\in\mathfrak{F}^n_R}b^n_{\lambda l}=0\right\}.$$

If the sum over $\mathfrak{F}^n_R$ can be replaced by $\mathfrak{E}^n_R$, we are done. Since

$$\mathfrak{F}^n_R\subset\left(\mathfrak{F}^n_R\cap\overline{\mathfrak{E}^n_{R+\gamma}}\right)\cup\mathfrak{E}^n_{R+\gamma}$$

holds,

$$\sum_{(\lambda,l)\in\mathfrak{F}^n_R}b^n_{\lambda l}\le\sum_{(\lambda,l)\in\mathfrak{F}^n_R\cap\overline{\mathfrak{E}^n_{R+\gamma}}}b^n_{\lambda l}+\sum_{(\lambda,l)\in\mathfrak{E}^n_{R+\gamma}}b^n_{\lambda l}.$$

We show the first term of the right hand side is negligible for any $\gamma>0$:

$$\begin{aligned}\sum_{(\lambda,l)\in\mathfrak{F}^n_R\cap\overline{\mathfrak{E}^n_R}}b^n_{\lambda l}&\le 2^{-n(R+\gamma)}\left|\left\{(\lambda,l)\ ;d_\lambda\le 2^{nR}\ \right\}\right|\\&=\ 2^{-n(R+\gamma)}\sum_{\lambda:d_\lambda\le 2^{nR}}d_\lambda\\&\le 2^{-n(R+\gamma)}\cdot 2^{nR}\cdot(n+1)^d\\&=(n+1)^d\,2^{-n\gamma}.\end{aligned}$$

Therefore,

$$\begin{aligned}E_d\left(\{\rho^n\}_{n=1}^{\infty}\right)&\ge\sup_R\left\{R\ ;\lim_{n\to\infty}\left(\sum_{(\lambda,l)\in\mathfrak{E}^n_{R+\gamma}}b^n_{\lambda l}+(n+1)^d\,2^{-n\gamma}\right)=0\right\}\\&=\sup_R\left\{R\ ;\ \lim_{n\to\infty}\sum_{(\lambda,l)\in\mathfrak{E}^n_{R+\gamma}}b^n_{\lambda l}\le\epsilon\right\}\\&=\text{p}-\varliminf_{n\to\infty}\frac{-1}{n}\log b^n_{\lambda l}-\gamma.\end{aligned}$$

Since this holds for all γ, the lemma is proven. ■

Lemma 8 *If Alice's view of* $|\psi^n\rangle$ *is the same as* ρ^n*, i.e.,*

$$\mathrm{tr}_{\mathcal{H}_B} |\psi^n\rangle \langle\psi^n| = \rho^n,$$

$$E_d\left(\{\rho^n\}_{n=0}^{\infty}\right) \leq E_d\left(\{|\psi^n\rangle\}_{n=0}^{\infty}\right)$$

Proof. We prove that ρ^n can be made from $|\psi^n\rangle$ by a local operation. Let $|\psi'\rangle \in (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n} \otimes \mathcal{K}$ be a purification of ρ^n. Since Alice's view of $|\psi'\rangle$ and $|\psi^n\rangle$ are the same, $|\psi'\rangle$ is mapped to $|\psi^n\rangle$ by a local isometry acting on $\mathcal{H}_B^{\otimes n} \otimes \mathcal{K}$. ■

Theorem 9 *If* ρ^n *is supported on the symmetric subspace of* $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$*,*

$$\begin{aligned} E_d\left(\{\rho^n\}_{n=0}^{\infty}\right) &= \text{p}-\varliminf_{n\to\infty} \frac{-1}{n} \log b_{\lambda l}^n \\ &= \text{p}-\varliminf_{n\to\infty} \frac{-1}{n} \log c_{\lambda k l}^n. \end{aligned}$$

A remarkable point is that the optimal rate can be achieved without knowing ρ^n, as is indecated in lemma 7. This is a natural generalization of universal entanglement concentration in [12].

Proof. Due to [6] and the above lemma,

$$E_d\left(\{\rho^n\}_{n=0}^{\infty}\right) \leq \text{p}-\varliminf_{n\to\infty} \frac{-1}{n} \log c_{\lambda k l}^n.$$

Due to lemma 7,

$$\text{p}-\varliminf_{n\to\infty} \frac{-1}{n} \log b_{\lambda l}^n \leq \text{p}-\varliminf_{n\to\infty} \frac{-1}{n} \log c_{\lambda k l}^n.$$

Hence, our task is only to show the opposite inequality:

$$\begin{aligned} &\text{p}-\varliminf_{n\to\infty} \frac{-1}{n} \log b_{\lambda l}^n \\ &= \text{p}-\varliminf_{n\to\infty} \frac{-1}{n} \left[\log c_{\lambda k l}^n - \log\left(c_{\lambda k l}^n / b_{\lambda l}^n\right)\right] \\ &\geq \text{p}-\varliminf_{n\to\infty} \frac{-1}{n} \log c_{\lambda k l}^n + \text{p}-\varliminf_{n\to\infty} \frac{1}{n} \log\left(c_{\lambda k l}^n / b_{\lambda l}^n\right) \\ &= \text{p}-\varliminf_{n\to\infty} \frac{-1}{n} \log c_{\lambda k l}^n - \text{p}-\varlimsup_{n\to\infty} \frac{-1}{n} \log\left(c_{\lambda k l}^n / b_{\lambda l}^n\right) \\ &= \text{p}-\varliminf_{n\to\infty} \frac{-1}{n} \log c_{\lambda k l}^n, \end{aligned}$$

where the last equality is due to (14). ■

6 Strong converse

The strong converse property for entanglement dilution is defined as follows: If $\mathrm{F}^{2^{nR}}(\rho^n) \to 0$ occurs for all $R < E_c(\{\rho^n\}_{n=0}^{\infty})$, we say that $\{\rho^n\}_{n=0}^{\infty}$ has strong converse property for dilution

Similarly, the strong converse property for entanglement distillation is defined as follows. Denote by $\mathrm{F}_d^D(\rho)$ the optimal fidelity of making the maximally entangled state with Schmidt rank D from ρ by LOCC. $\{\rho^n\}_{n=0}^{\infty}$ is said to have strong converse property for distillation if $\mathrm{F}_d^{2^{nR}}(\rho^n) \to 0$ occurs for all $R > E_d(\{\rho^n\}_{n=0}^{\infty})$.

Theorem 10 *Suppose ρ^n is a symmetric state. Then, the following three conditions are equivalent. (i) Entanglement dilution of $\{\rho^n\}_{n=0}^{\infty}$ has strong converse property. (ii) Entanglement distillation from $\{\rho^n\}_{n=0}^{\infty}$ has strong converse property. (iii) $E_c(\{\rho^n\}_{n=0}^{\infty}) = E_d(\{\rho^n\}_{n=0}^{\infty})$.*

Proof. First we prove (i)$\Leftarrow$(iii). Combination of (8) and (11) yields

$$\mathrm{F}^{2^{nR}}(\rho^n) \le (n+1)^d\, 2^{-n\gamma} + \sum_{(\lambda,l)\in\mathfrak{E}^n_{R+\gamma}} b^n_{\lambda l}.$$

Hence, if

$$R+\gamma < \mathrm{p}\text{-}\varliminf_{n\to\infty} \frac{-1}{n}\log b^n_{\lambda l} = E_d(\{\rho^n\}_{n=0}^{\infty}),$$

the last end asymptotically vanishes. Since $\gamma > 0$ is arbitrary, we obtain (i)$\Leftarrow$(iii). On the other hand, if (i) holds, the entanglement dilution protocol mentioned right before the theorem 5 also can achieve only asymptotically vanishing fidelity with $R < E_c(\{\rho^n\}_{n=0}^{\infty})$:

$$\lim_{n\to\infty} \sum_{(\lambda,l)\in\mathfrak{F}^n_{R^n}} b^n_{\lambda l} = 0,$$

where R^n is defined by (9). Due to (12), this implies

$$\lim_{n\to\infty} \sum_{(\lambda,l)\in\mathfrak{E}^n_{R^n}} b^n_{\lambda l} = 0.$$

Since $R^n \to R$ as $n\to\infty$ due to (10), this implies

$$R \le \mathrm{p}\text{-}\varliminf_{n\to\infty} \frac{-1}{n}\log b^n_{\lambda l} = E_d(\{\rho^n\}_{n=0}^{\infty}).$$

Therefore, we have (i)$\Rightarrow$(iii). Next, we suppose that (ii) holds. Then, with $R > E_d(\{\rho^n\}_{n=0}^{\infty})$, the protocol in the proof of lemma 7 can achieve only asymptotically vanishing fidelity :

$$\lim_{n\to\infty} \sum_{(\lambda,l)\notin\mathfrak{F}^n_{R^n}} b^n_{\lambda l} = 0.$$

Observe

$$
\begin{aligned}
1-\sum_{(\lambda,l)\notin\mathfrak{F}^n_{Rn}} b^n_{\lambda l} &= \sum_{(\lambda,l)\in\mathfrak{F}^n_{Rn}} b^n_{\lambda l} \\
&= \mathrm{F}^{2^{nR^n}}(\sigma^n) \\
&\leq \mathrm{F}^{2^{nR^n}}(\rho^n) \\
&\leq \mathrm{F}^{2^{nR}}(\rho^n) \\
&\leq (n+1)^d\,2^{-n\gamma}+\sum_{(\lambda,l)\in\mathfrak{E}^n_{R+\gamma}} b^n_{\lambda l},
\end{aligned}
$$

where the inequality in the third, fourth, and the last line is due to (8), (10), and (11), respectively. Therefore, we have

$$
\lim_{n\to\infty}\sum_{(\lambda,l)\in\mathfrak{E}^n_{R+\gamma}} b^n_{\lambda l}=1
$$

for any $\gamma>0$, or equivalently,

$$
R\geq \mathrm{p-}\overline{\lim_{n\to\infty}}\frac{-1}{n}\log b^n_{\lambda l}=E_c\left(\{\rho^n\}_{n=0}^{\infty}\right).
$$

Therefore, we have (ii)$\Rightarrow$(iii). Finally, we show (ii)$\Leftarrow$(iii). Let $|\psi^n\rangle$ be a purification of ρ^n, with all the ancilla at Bob's hand. Obviously,

$$
\mathrm{F}_d^{2^{nR}}\left(|\psi^n\rangle\right)\geq \mathrm{F}_d^{2^{nR}}\left(\rho^n\right).
$$

Suppose $R>\mathrm{p-}\overline{\lim}_{n\to\infty}\frac{-1}{n}\log c^n_{\lambda kl}$. Then, [6] had shown that $\mathrm{F}_d^{2^{nR}}\left(|\psi^n\rangle\right)\to 0$.

$$
\mathrm{p-}\overline{\lim_{n\to\infty}}\frac{-1}{n}\log c^n_{\lambda kl}=E_c\left(\{\rho^n\}_{n=0}^{\infty}\right)=E_d\left(\{\rho^n\}_{n=0}^{\infty}\right)
$$

holds by assumption, this implies strong converse for distillation. ■

7 Output of an optimal cloning machine (1)

In this and next section, we study the ouput states of cloning machines. They are, if optimally desined for pure input states, mixed symmetric states.

In this section, we suppose that the Schmidt basis of the given pure state, except for its phases, are known i.e.,

$$
|\phi\rangle=\sum_{i=1}^{d}\sqrt{p_i}e^{\sqrt{-1}\theta_i}\,|i\rangle\,|i\rangle\,,
$$

where $\boldsymbol{p}=(p_1,\cdots,p_d)$ and θ_i $(i=1,\cdots,d)$ are unknown. The final state of Its optimal n to m cloning machine is

$$\sum_{\boldsymbol{n},\boldsymbol{m}} \alpha_{\boldsymbol{m},\boldsymbol{n}} \left|\boldsymbol{m}\right\rangle \otimes \left|R_{\boldsymbol{m}-\boldsymbol{n}}\right\rangle ,$$

where

$$\left|\boldsymbol{m}\right\rangle = \sqrt{\frac{\prod_{k=1}^{d} m_k!}{m!}} \sum_{\#\{\kappa\,;\, i_\kappa = k\} = m_k} \bigotimes_{\kappa=1}^{m} \left|i_\kappa\right\rangle \left|i_\kappa\right\rangle \ ,$$

$$\alpha_{\boldsymbol{m},\boldsymbol{n}} = \sqrt{\frac{(m-n)!\,(n+d-1)!}{(m+d-1)!}} \prod_{k=1}^{d} \sqrt{\frac{m_k!}{n_k!\ (m_k-n_k)!}} \sqrt{\frac{n!}{\prod_{k=1}^{d} n_k!}} p_k^{n_k} e^{\sqrt{-1}\theta_k}$$

and $\{|R_{\boldsymbol{j}}\rangle\}$ is an orthonormal basis of the internal state of the optimal cloning machine [2]. Tracing out the internal state of the machine, we obtain the output state, which is denoted by $\rho_1^{n,m}$. Below, we denote by $H(\boldsymbol{p})$ the Shannon entropy of the probability distribution $\boldsymbol{p}$.

Theorem 11

$$E_c\left(\left\{\rho_1^{m/r,m}\right\}_{m=1}^{\infty}\right) = E_d\left(\left\{\rho_1^{m/r,m}\right\}_{m=1}^{\infty}\right) = H(\boldsymbol{p}). \tag{15}$$

An important consequence of this is that the strong converse holds for $\left\{\rho_1^{m/r,m}\right\}_{m=1}^{\infty}$.

Also, the real m copies and optimal clone are the same in entanglement quantities. (Recall all the reasonable entanglement measures lies between E_d and E_c.) This is rather surprising since $\mathrm{F}\left(\rho_1^{m/r,m}, |\phi\rangle^{\otimes m}\right) \approx r^d$, and these two states are not so close.

However, with closer look, entanglement of $\rho_1^{m/r,m}$ and $|\phi\rangle^{\otimes m}$ are somewhat different. More concretely, they differ in error exponent of entanglement dilution. Below, $h(x) := -x\log x - (1-x)\log(1-x)$.

Theorem 12 *If $R > H(\boldsymbol{p})$,*

$$\begin{aligned}
&\lim_{m\to\infty} \frac{-1}{m} \log\left\{1 - \mathrm{F}^{2^{mR}}\left(\rho_1^{m/r,m}\right)\right\} \\
&= \min_{\boldsymbol{q}:\, H(\boldsymbol{q})\geq R}\ \min_{\boldsymbol{q}': q_i' \leq r q_i} \left\{ h\left(\frac{1}{r}\right) - \sum_{i=1}^{d} q_i\, h\left(\frac{q_i'}{r q_i}\right) + \frac{1}{r} D\left(\boldsymbol{q}' || \boldsymbol{p}\right)\right\}.
\end{aligned} \tag{16}$$

Equivalently, if $\lim_{m\to\infty} \frac{-1}{m} \log\left\{1 - \mathrm{F}^{2^{mR}}\left(\rho_1^{m/r,m}\right)\right\} \geq \eta$, *we at least need following ebits of maximally entangled state:*

$$\max\left\{ H(\boldsymbol{q})\ ;\ \min_{\boldsymbol{q}': q_i' \leq r q_i} \left\{ h\left(\frac{1}{r}\right) - \sum_{i=1}^{d} q_i\, h\left(\frac{q_i'}{r q_i}\right) + \frac{1}{r} D\left(\boldsymbol{q}' || \boldsymbol{p}\right)\right\} \geq \eta \right\}.$$

Observe that (16) is smaller than or equal to

$$\frac{1}{r} \min_{\boldsymbol{q} \colon H(\boldsymbol{q}) \geq R} D\left(\boldsymbol{q} || \boldsymbol{p}\right).$$

It is known that the exponent for $|\phi\rangle^{\otimes m}$,

$$\lim_{m \to \infty} \frac{-1}{m} \log \left\{ 1 - \mathrm{F}^{2^{mR}} \left(|\phi\rangle^{\otimes m} \right) \right\} = \min_{\boldsymbol{q} \colon H(\boldsymbol{q}) \geq R} D\left(\boldsymbol{q} || \boldsymbol{p}\right)$$

(see [7]). Therefore, (16) is smaller than or equal to the exponent for $|\phi\rangle^{\otimes (m/r)}$, or the input of the cloning machine.

7.1 Proof of theorem 11

The eigenvectors of the reduced density matrix $\mathrm{tr}_{\mathcal{H}_B^{\otimes m}} \rho_1^{n,m}$ are

$$|i_1 \cdots i_m\rangle := \bigotimes_{\kappa=1}^{m} |i_\kappa\rangle$$

with corresponding eigenvalues

$$\frac{\prod_{k=1}^{d} m_k!}{m!} \sum_{\boldsymbol{n}} \left|\alpha_{\boldsymbol{m},\boldsymbol{n}}\right|^2,$$

where $\#\{\,\kappa\,;\, i_\kappa = k\} = m_k$. Each eigenvalue has $m!/\prod_{k=1}^{d} m_k!$ folds degeneracy.

Due to theorem 6, $R \geq E_c\left(\{\rho_1^{n,rn}\}_{n=1}^{\infty}\right)$ holds if and only if

$$\lim_{n \to \infty} \sum_{\boldsymbol{m} \notin \mathfrak{G}_R^m} \sum_{\boldsymbol{n}} \left|\alpha_{\boldsymbol{m},\boldsymbol{n}}\right|^2 = 0$$

holds, where

$$\mathfrak{G}_R^m := \left\{ \boldsymbol{m} \;;\; \frac{\prod_{k=1}^{d} m_k\,!}{m!} \sum_{\boldsymbol{n}} \left|\alpha_{\boldsymbol{m},\boldsymbol{n}}\right|^2 \geq 2^{-mR} \right\}$$

Letting $\boldsymbol{n}^* = \arg\max_{\boldsymbol{n}} \left|\alpha_{\boldsymbol{m},\boldsymbol{n}}\right|^2$, we have

$$\begin{aligned} &\sum_{\boldsymbol{m} \notin \mathfrak{G}_R^m} \sum_{\boldsymbol{n}} \left|\alpha_{\boldsymbol{m},\boldsymbol{n}}\right|^2 \\ &\leq poly\,(n) \times \max_{\boldsymbol{m} \notin \mathfrak{G}_R^m} 2^{-m\left\{ h\left(\frac{1}{r}\right) - \sum_{k=1}^{d} \frac{m_k}{m} h\left(\frac{n_k^*}{m_k}\right) + \frac{1}{r} D\left(\frac{\boldsymbol{n}^*}{n} || \boldsymbol{p}\right) \right\}} \end{aligned}$$

Observe

$$h\left(\frac{1}{r}\right) - \sum_{k=1}^{d} \frac{m_k}{m} h\left(\frac{n_k^*}{m_k}\right) \geq 0$$

and

$$\frac{1}{r}D\left(\frac{\boldsymbol{n}^*}{n}||\,\boldsymbol{p}\right)\geq 0$$

holds, and the identity holds if and only if $\boldsymbol{m}=r\boldsymbol{n}^*$ and $\boldsymbol{n}^*=n\boldsymbol{p}$, respectively. Hence, $R\geq E_c\left(\left\{\rho_1^{m/r,\,m}\right\}_{m=1}^{\infty}\right)$ holds if at least one of the equality does not hold, or equivalently

$$\boldsymbol{m}=rn\,\boldsymbol{p}=m\boldsymbol{p}\in\mathfrak{G}_R^m$$

holds, or equivalently,

$$\begin{aligned}
R &\geq \frac{1}{m}\log\frac{m!}{\prod_{k=1}^{d}m_k\,!}+\frac{-1}{m}\log\sum_{\boldsymbol{n}}\left|\alpha_{\boldsymbol{m},\boldsymbol{n}}\right|^2\\
&\geq H\left(\boldsymbol{p}\right)+h\left(\frac{1}{r}\right)-\sum_{k=1}^{d}\frac{m_k}{m}h\left(\frac{n_k^*}{m_k}\right)+\frac{1}{r}D\left(\frac{\boldsymbol{n}^*}{n}||\,\boldsymbol{p}\right)+\frac{C}{m}\log m\\
&= H\left(\boldsymbol{p}\right)+\frac{C}{m}\log m,
\end{aligned}$$

holds for all m. If this holds as $m\to\infty$, $R\geq E_c\left(\left\{\rho_1^{m/r,m}\right\}_{m=1}^{\infty}\right)$. Therefore, we obtain

$$E_c\left(\left\{\rho_1^{m/r,m}\right\}_{m=1}^{\infty}\right)\leq H\left(\boldsymbol{p}\right).$$

Due to theorem 9. $R\leq E_d\left(\left\{\rho_1^{m/r,m}\right\}_{m=1}^{\infty}\right)$ holds if and only if

$$\lim_{n\to\infty}\sum_{\boldsymbol{m}\in\mathfrak{G}_R^m}\sum_{\boldsymbol{n}}\left|\alpha_{\boldsymbol{m},\boldsymbol{n}}\right|^2=0.$$

The left hand side can be evaluated in the same way as above, and we can easily see that the condition is true if

$$\boldsymbol{m}=rn\,\boldsymbol{p}=m\boldsymbol{p}\in\mathfrak{G}_R^m$$

holds. This is equivalent to

$$R\leq H\left(\boldsymbol{p}\right)+\frac{C}{m}\log n.$$

If this holds as $n\to\infty$, $R\leq E_d\left(\left\{\rho_1^{m/r,m}\right\}_{m=1}^{\infty}\right)$. Therefore, we obtain

$$H\left(\boldsymbol{p}\right)\leq E_d\left(\left\{\rho_1^{m/r,m}\right\}_{m=1}^{\infty}\right).$$

After all, we have

$$H\left(\boldsymbol{p}\right)\leq E_d\left(\left\{\rho_1^{m/r,m}\right\}_{m=1}^{\infty}\right)\leq E_c\left(\left\{\rho_1^{m/r,m}\right\}_{m=1}^{\infty}\right)\leq H\left(\boldsymbol{p}\right).$$

Therefore, we have the theorem.

7.2 Proof of theorem 12

We only have to prove (16).

To prove "$\geq$", we consider the entanglement dilution protocol mentioned right before theorem 5.

$$
\begin{aligned}
1-\mathrm{F}^{2^{mR}}\left(\rho_1^{m/r,m}\right) &\leq \sum_{(\lambda,l)\notin\mathfrak{F}^m_{Rm}} b^m_{\lambda l} \\
&= \sum_{(\lambda,l)\notin\mathfrak{F}^m_{Rm}} \sum_{k=1}^{\dim\mathcal{U}_\lambda} c^m_{\lambda k l} \\
&\leq \sum_{(\lambda,l)\notin\mathfrak{F}^m_{Rm}} \dim\mathcal{U}_\lambda \times \max_k c^m_{\lambda k l} \\
&\leq \mathit{poly}\,(n) \times \max_{(\lambda,l)\notin\mathfrak{F}^m_{Rm}} d_\lambda \max_k c^m_{\lambda k l} \\
&\leq \mathit{poly}\,(n) \times \max_{\lambda: H\left(\frac{\lambda}{m}\right)\geq R-\gamma} d_\lambda \max_k c^m_{\lambda k l} \\
&\leq \mathit{poly}\,(n) \times \max_{\lambda: H\left(\frac{\lambda}{m}\right)\geq R-\gamma} 2^{mH\left(\frac{\lambda}{m}\right)} \max_k c^m_{\lambda k l}\ ,
\end{aligned}
\tag{17}
$$

whee γ is an arbitrary positive constant.

Apply random $U_A^{\otimes m}\otimes U_B^{\otimes m}$ to $\rho_1^{n,m}$, and denote by σ_1^m the product, which is in the form of (7). Since this operation is LOCC,

$$
\begin{aligned}
&1-\mathrm{F}^{2^{mR}}\left(\rho_1^{m/r,m}\right) \geq 1-\mathrm{F}^{2^{mR}}\left(\sigma_1^m\right) \\
&= \sum_{(\lambda,l)\notin\mathfrak{F}^m_{Rm}} b^m_{\lambda l} \geq \sum_{(\lambda,l)\notin\mathfrak{F}^m_{R}} b^m_{\lambda l} \\
&\geq \max_{(\lambda,l)\notin\mathfrak{F}^m_{R}} d_\lambda b^m_{\lambda\, l} \\
&\geq \max_{(\lambda,l)\notin\mathfrak{F}^m_{R}} d_\lambda \sum_k c^m_{\lambda k l} \\
&\geq \max_{(\lambda,l)\notin\mathfrak{F}^m_{R}} d_\lambda \max_k c^m_{\lambda k l} \\
&\geq \max_{\lambda: H\left(\frac{\lambda}{m}\right)\geq R+\gamma'} d_\lambda \max_k c^m_{\lambda k l} \\
&\geq \frac{1}{\mathit{poly}\,(m)} \max_{\lambda: H\left(\frac{\lambda}{m}\right)\geq R+\gamma'} 2^{mH\left(\frac{\lambda}{m}\right)} \max_k c^m_{\lambda k l},
\end{aligned}
\tag{18}
$$

where γ' is an arbitrary positive constant. Letting $\gamma\to 0$ and $\gamma'\to 0$, combination of (17) and (18) yields

$$
\frac{-1}{m}\log\left[1-\mathrm{F}^{2^{mR}}\left(\rho_1^{m/r,m}\right)\right] = \min_{\lambda: H\left(\frac{\lambda}{m}\right)\geq R} \min_k \left[\frac{-1}{m}\log c^m_{\lambda k l} - H\left(\frac{\lambda}{m}\right)\right] + o\,(1)\,.
$$

A key observation is:

$$c_{\lambda kl}^{m}=\frac{\prod_{i=1}^{d}\lambda_{i}^{k}\,!}{\lambda^{k}!}\sum_{\boldsymbol{\mu}}\left|\alpha_{\lambda^{k},\boldsymbol{\mu}}\right|^{2}$$

holds for a λ^{k} with $\lambda^{k}\prec\lambda$. This is because: (i) the eigenvectors are in the form of $\bigotimes_{\kappa}|i_{\kappa}\rangle$. (ii) the eigenvalue depends only on $m_{j}=\#\{\kappa\,;\,i_{\kappa}=j\}$. Therfore,

$$\begin{aligned}
&\min_{\lambda:H\left(\frac{\lambda}{m}\right)\geq R}\min_{k}\left[\frac{-1}{m}\log c_{\lambda kl}^{m}-H\left(\frac{\lambda}{m}\right)\right]\\
&=\min_{\lambda:H\left(\frac{\lambda}{m}\right)\geq R}\min_{\lambda':\lambda'\prec\lambda}\left[\frac{-1}{m}\log\frac{\prod_{i=1}^{d}\lambda_{i}'\,!}{\lambda'!}\sum_{\boldsymbol{\mu}}\left|\alpha_{\lambda',\boldsymbol{\mu}}\right|^{2}-H\left(\frac{\lambda}{m}\right)\right]\\
&=\min_{\lambda:H\left(\frac{\lambda}{m}\right)\geq R}\min_{\lambda':\lambda'\prec\lambda}\min_{\mu:\mu_{i}\leq\lambda_{i}'}\left[\begin{array}{c}H\left(\frac{\lambda'}{m}\right)-H\left(\frac{\lambda}{m}\right)+h\left(\frac{1}{r}\right)-\sum_{k=1}^{d}\frac{\lambda_{i}'}{m}h\left(\frac{\mu_{i}}{\lambda_{i}'}\right)\\+\frac{1}{r}D\left(\frac{\boldsymbol{\mu}}{n}\|\,\boldsymbol{p}\right)\end{array}\right]+o\left(1\right).
\end{aligned}\tag{19}$$

Since $H\left(\cdot\right)$ is Shur concave,

$$\min_{\lambda:H\left(\frac{\lambda}{m}\right)\geq R}\min_{\lambda':\lambda'\prec\lambda}\geq\min_{\lambda,\lambda':H\left(\frac{\lambda'}{m}\right)\geq H\left(\frac{\lambda}{m}\right)\geq R}.$$

Observe λ appears only in $-H\left(\frac{\lambda}{m}\right)$. SInce

$$-H\left(\frac{\lambda}{m}\right)\geq-H\left(\frac{\lambda'}{m}\right),$$

the optimal λ equals λ'. Therefore, (19) is lowerbounded by the right hand side of (16) except for $o\left(1\right)$-terms. On the other hand, by simply substituting $\lambda'=\lambda$, we can prove (19) is upperbounded by the right hand side of (16).

8 Output of an optimal cloning machine (2)

Here, we consider the case where a given state can be an arbitrary pure state. Our conjecture is that the entanglement cost is again $H\left(\boldsymbol{p}\right)$. However, we can only show that $H\left(\boldsymbol{p}\right)$ is an upperbound.

Letting

$$\alpha_{\tilde{\boldsymbol{m}},\tilde{\boldsymbol{n}}}:=\sqrt{\frac{(m-n)!\,(n+d^{2}-1)!}{(m+d^{2}-1)!}}\prod_{k,l=1}^{d}\sqrt{\frac{\tilde{m}_{k,l}!}{\tilde{n}_{k,l}!\,\left(\tilde{m}_{k,l}-\tilde{n}_{k,l}\right)!}\frac{n!}{\prod_{k,l=1}^{d}\tilde{n}_{k,l}!}p_{k}^{\tilde{n}_{k,k}}}\delta_{k,l},$$

and $\left|R_{\tilde{\boldsymbol{m}}-\tilde{\boldsymbol{n}}}\right\rangle$ be the internal state of the cloning machine, the final state of optimal cloning machine is given as follows [2].

$$\sum_{\tilde{\boldsymbol{m}},\tilde{\boldsymbol{l}}}\alpha_{\tilde{\boldsymbol{m}},\tilde{\boldsymbol{n}}}\sqrt{\frac{\prod_{j,k=1}^{d}\tilde{m}_{j,k}!}{m!}}\sum_{\#\{(\kappa,\mu):(i_{\kappa},i_{\mu})=(j,k)\}=\tilde{m}_{j,k}}\bigotimes_{\kappa,\mu=1}^{m}|i_{\kappa}\rangle\,|i_{\mu}\rangle\otimes\left|R_{\tilde{\boldsymbol{m}}-\tilde{\boldsymbol{n}}}\right\rangle.$$

Denote by $\rho_2^{n,m}$ the state after tracing out the internal state of cloning machine. $\rho_2^{n,m}$ is probability mixture of

$$\frac{1}{\sqrt{\beta_{\tilde{\boldsymbol{n}}}}}\sum_{\tilde{\boldsymbol{m}}}\alpha_{\tilde{\boldsymbol{m}},\tilde{\boldsymbol{n}}}\sqrt{\frac{\prod_{j,k=1}^{d}\tilde{m}_{j,k}!}{m!}}\sum_{\#\{(\kappa,\mu):(i_\kappa,i_\mu)=(j,k)\}=\tilde{m}_{j,k}}\bigotimes_{\kappa,\mu=1}^{m}|i_\kappa\rangle|i_\mu\rangle,$$

with the probability $\beta_{\tilde{\boldsymbol{n}}}$, where

$$\beta_{\tilde{\boldsymbol{l}}}=\sum_{\tilde{\boldsymbol{m}}}|\alpha_{\tilde{\boldsymbol{m}},\tilde{\boldsymbol{n}}}|^2.$$

Now we apply

$$\bigotimes_{\kappa=1}^{m}\sum_{i,j=1}^{d}e^{\sqrt{-1}\omega_{i,\kappa}^{A}}|i\rangle\langle i|\otimes e^{\sqrt{-1}\omega_{j,\kappa}^{B}}|j\rangle\langle j|,$$

where $\omega_{i,\kappa}^{A}$, $\omega_{j,\kappa}^{B}$ are chosen independently randomly. After the application of this operation, Bob's local view will have the density matrix with the eigenvector $\bigotimes_{\kappa=1}^{m}|i_\kappa\rangle$ and the corresponding eigenvalue

$$\sum_{\sum_{k=1}^{d}\tilde{m}_{j,k}=m_j^A}\frac{\prod_{j,k=1}^{d}\tilde{m}_{j,k}!}{m!}|\alpha_{\tilde{\boldsymbol{m}},\tilde{\boldsymbol{n}}}|^2,$$

where $m_j^A=\#\{\kappa;i_\kappa=j\}$. Each eigenvalue has $m!/\prod_{j=1}^{d}m_j^A!$ folds degeneracy. To compute these, it is easier to apply dephasing at both parties first, and take partial trace later.

Lemma 13 *Let $\boldsymbol{q}^n$ be the spectrum of the reduced density matrix ρ^n. Then,*

$$\mathrm{p}-\overline{\lim_{n\to\infty}}\frac{-1}{n}\log q_i^n$$

is decreasing by application of the above operation.

Proof. Consider a pure state $|\psi^n\rangle$ with the Schmidt coefficients $\boldsymbol{q}^n$. Then, $E_c(\{|\psi^n\rangle\}_{n=1}^{\infty})=\mathrm{p}-\overline{\lim}_{n\to\infty}\frac{-1}{n}\log q_i^n$. Therefore, $\mathrm{p}-\overline{\lim}_{n\to\infty}\frac{-1}{n}\log q_i^n$ has to be Shur concave, and should be monotone with respect to the probabilistic unitary. ■

Remark 14 *Since the above dephasing operation is LOCC, the entanglement cost of the resultant state is a lowerboud of it of the optimal clone. However, this state is not supported on the symmetric subspace anymore, and we cannot apply our formula.*

Hence, letting

$$\mathfrak{H}_R^n = \left\{ \left(\tilde{\boldsymbol{m}}^A , \tilde{\boldsymbol{n}} \right) ; \sum_{\tilde{\boldsymbol{m}}: \sum_{k=1}^d \tilde{m}_{j,k} = m_j^A} \frac{\prod_{j,k=1}^d \tilde{m}_{j,k}!}{m!} \left| \alpha_{\tilde{\boldsymbol{m}},\tilde{\boldsymbol{n}}} \right|^2 \geq 2^{-nR} \right\},$$

and denoting by $\tilde{\boldsymbol{p}}$ the probability distribution $p_i \delta_{i,j}$ over the set $\{(i,j); i,j = 1, \cdots d\}$, $R \geq E_c \left(\left\{ \rho_2^{m/r\,,\,m} \right\}_{m=1}^\infty \right)$ holds is the following sum goes to 0.

$$\begin{aligned} &\sum_{\left(\tilde{\boldsymbol{m}}^A , \tilde{\boldsymbol{n}} \right) \notin \mathfrak{h}_R^n} \frac{m!}{\prod_{j=1}^d m_j^A!} \sum_{\sum_{k=1}^d \tilde{m}_{j,k} = m_j^A} \frac{\prod_{j,k=1}^d \tilde{m}_{j,k}!}{m!} \left| \alpha_{\tilde{\boldsymbol{m}},\tilde{\boldsymbol{n}}} \right|^2 \qquad (20) \\ &\leq poly\,(m) \times \max_{\tilde{\boldsymbol{m}}^A , \tilde{\boldsymbol{n}}, \tilde{\boldsymbol{m}}} 2^{-m \left\{ -H\left(\frac{\boldsymbol{m}^A}{m} \right) + H\left(\frac{\tilde{\boldsymbol{m}}}{m} \right) + h\left(\frac{1}{r} \right) - \sum_{k,l} \frac{\tilde{m}_{k,l}}{m} h\left(\frac{\tilde{n}_{k,l}}{\tilde{m}_{k,l}} \right) + \frac{1}{r} D\left(\frac{\tilde{\boldsymbol{n}}}{n} || \tilde{\boldsymbol{p}} \right) \right\}}, \end{aligned}$$

where the maximization is taken over all $\tilde{\boldsymbol{m}}^A, \tilde{\boldsymbol{n}}, \tilde{\boldsymbol{m}}$ with

$$\left(\tilde{\boldsymbol{m}}^A , \tilde{\boldsymbol{n}} \right) \notin \mathfrak{h}_R^n,\ \tilde{n}_{jk} \leq \tilde{m}_{jk},\ \sum_{k=1}^d \tilde{m}_{j,k} = m_j^A. \qquad (21)$$

Observe that

$$\begin{aligned} -H\left(\frac{\boldsymbol{m}^A}{m} \right) + H\left(\frac{\tilde{\boldsymbol{m}}}{m} \right) &\geq 0, \\ h\left(\frac{1}{r} \right) - \sum_{k,l} \frac{\tilde{m}_{k,l}}{m} h\left(\frac{\tilde{n}_{k,l}}{\tilde{m}_{k,l}} \right) &\geq 0, \\ \frac{1}{r} D\left(\frac{\tilde{\boldsymbol{n}}}{n} || \tilde{\boldsymbol{p}} \right) &\geq 0. \end{aligned}$$

The identity in each inequality holds if and only if

$$\begin{aligned} \frac{\boldsymbol{m}^A}{m} &= \frac{\tilde{\boldsymbol{m}}}{m}, \\ \frac{\tilde{\boldsymbol{n}}}{n} &= \frac{\tilde{\boldsymbol{m}}}{m}, \\ \tilde{\boldsymbol{p}} &= \frac{\tilde{\boldsymbol{n}}}{n}, \end{aligned}$$

holds, respectively. Hence, the right hand side of (20) converges to 0 if one of these does not hold, or equivalently,

$$\tilde{\boldsymbol{m}} = r\tilde{\boldsymbol{n}} = m\tilde{\boldsymbol{p}} \in \mathfrak{h}_R^n,$$

or equivalently,

$$
\begin{aligned}
R &\geq H(\tilde{\boldsymbol{p}}) + h\left(\frac{1}{r}\right) - \sum_{k,l} \frac{\tilde{m}_{k,l}}{m} h\left(\frac{1}{r}\right) + \frac{1}{r} D(\tilde{\boldsymbol{p}} \| \tilde{\boldsymbol{p}}) + \frac{C}{m}\log m \\
&= H(\tilde{\boldsymbol{p}}) + \frac{C}{m}\log m.
\end{aligned}
$$

If this holds as $m \to \infty$, $E_c\left(\left\{\rho_2^{m/r\,,\,m}\right\}_{m=1}^{\infty}\right) \leq R$. Therefore, we have

$$
E_c\left(\left\{\rho_2^{m/r\,,\,m}\right\}_{m=1}^{\infty}\right) \leq H(\tilde{\boldsymbol{p}}) = H(\boldsymbol{p}).
$$

9 Discussions

We first computed entanglement cost and distillable entanglement of non-i.i.d mixed state explicitly, and also gave general formula. We also have shown that universal entanglement concentration can be extended to arbitrary symmetric states.

Surprisingly, the real m copies and optimal clone under some assumption are the same in entanglement quantities. This is rather surprising since $F\left(\rho_1^{m/r,m}, |\phi\rangle^{\otimes m}\right) \approx r^d$, and these two states are not so close. However, with closer look, entanglement of $\rho_1^{m/r,m}$ and $|\phi\rangle^{\otimes m}$ are somewhat different. More concretely, they differ in error exponent of entanglement dilution and distillation. This motivate to use entanglement cost and distillable entanglement with restriction to error exponent. Such a measure had been closely studied in [9] for i.i.d. pure ensembles, and in [6] for general purestates. However, detailed analysis for mixed state ensembles, either i.i.d. or non-i.i.d, are still to be studied.

Another interesting open problem is the entanglement cost and distillable entanglement of optimal clone of totally unknown purestates. Are they also same as these of $|\phi\rangle^{\otimes m}$?

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A Group representation theory

Lemma 15 *Let U_g and U'_g be an irreducible representation of G on the finite-dimensional space $\mathcal{H}$ and $\mathcal{H}'$, respectively. We further assume that U_g and U'_g are not equivalent. If a linear operator A in $\mathcal{H}\oplus\mathcal{H}'$ is invariant by the transform $A \to U_g \oplus U'_g A U_g^* \oplus U_g^{'*}$ for any g, $\mathcal{H}A\mathcal{H}' = 0$ [3].*

Lemma 16 *(Shur's lemma [3]) Let U_g be as defined in lemma 15. If a linear map A in $\mathcal{H}$ is invariant by the transform $A \to U_g A U_g^*$ for any g, $A = c\mathbf{1}_{\mathcal{H}}$.*

B Representation of symmetric group and SU(d)

Due to [3], we have

$$\dim \mathcal{U}_\lambda = \frac{\prod_{i<j}(l_i - l_j)}{\prod_{i=1}^{d-1}(d-i)!}, \tag{22}$$

$$d_\lambda = \dim \mathcal{V}_\lambda = \frac{n!}{\prod_{i=1}^{d}(\lambda_i + d - i)!}\prod_{i<j}(l_i - l_j), \tag{23}$$

with $l_i := \lambda_i + d - i$. It is easy to show

$$\log \dim \mathcal{U}_\lambda \leq d^2 \log n. \tag{24}$$

Let $a_\lambda^\phi = \mathrm{Tr}\left\{\mathcal{W}_{\lambda,A}\left(\mathrm{Tr}_B|\phi\rangle\langle\phi|\right)^{\otimes n}\right\}$ and the formulas in the appendix of [8] says

$$\left|\frac{\log d_\lambda}{n} - \mathrm{H}\left(\frac{\lambda}{n}\right)\right| \leq \frac{d^2+2d}{2n}\log(n+d), \tag{25}$$

$$\sum_{\frac{\lambda}{n}\in\mathfrak{R}} a_\lambda^\phi \leq (n+1)^{d(d+1)/2}\exp\left\{-n\min_{\boldsymbol{q}\in\mathfrak{R}}\mathrm{D}\left(\boldsymbol{q}\|\boldsymbol{p}\right)\right\}, \tag{26}$$

where $\mathfrak{R}$ is an arbitrary closed subset.

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